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Asymptotics for multiple Meixner polynomials

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A B S T R A C T

We study the asymptotic behavior of multiple Meixner polynomials of first and second kind, respectively [6]. We use an algebraic function formulation for the solution of the equilibrium problem with constraint to describe their zero distribution. Moreover, analyzing the limiting behavior of the coefficients of the recurrence relations for Multiple Meixner polynomials we obtain the main term of their asymptotics.

Keywords: Discrete orthogonal polynomials, Multiple orthogonal polynomials n th-root asymptotics, Recurrence relations, Vector equilibrium with external field and constraint

1. Introduction

This paper deals with the asymptotics of polynomial sequences (P_n) defined by orthogonality relations with respect to a discrete measure μ (with finite moments)

$$\mu = \sum_{k=0}^N \rho(x) \delta_{x_k}, \quad \rho(x_k) = \rho_k > 0, \quad x_k \in \mathbb{R}, \quad N \in \mathbb{N} \cup \{+\infty\}, \quad (1.1)$$

which is a linear combination of Dirac measures at the points x_0, \dots, x_N . By \mathbb{N} we denote the set of all nonnegative integers.

The asymptotic theory of orthogonal polynomials with respect to discrete measures is not so widely developed as for absolutely continuous measures of orthogonality. It can be explained because the distribution of zeros of the polynomials orthogonal with respect to a discrete measure has an extra constraint. Indeed, between two neighboring mass points of the measure of orthogonality it cannot be situated more than one zero of the orthogonal polynomials (this fact follows from the interlacing property of orthogonal polynomials). This constraint plays an important role in the logarithmic potential description of the limiting zero distribution measure as well as of the main term of asymptotics.

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The logarithmic potential

$$\mathcal{P}^v(z) = - \int \ln |z - t| d\nu(t),$$

for probability measure $\nu_{P_n}(t)$ (zero counting measure), equally distributed in the zeros of polynomial P_n , is given by

$$\mathcal{P}^{\nu_{P_n}} = -\frac{1}{n} \ln |P_n|.$$

Thus, the potential of a limiting (when $n \rightarrow \infty$) zero distribution measure λ defines an *exponent of the main term* of the asymptotics for the polynomial sequence (P_n) . Such type of asymptotics are also called *weak asymptotics*. For the description of this limiting measure it is useful to formulate a minimization problem for the energy of the logarithmic potential. It is clear, that the constraint of the zero counting measure (which we pointed out above) leads to a problem of energy minimization in the class of probabilistic measures constrained by the weak limit of the mass points of the zero counting measure. For discrete orthogonal polynomials such an approach was suggested by Rakhmanov in [21] and extended later by Dragnev and Saff in [9]. For more details on the topic we refer to monographs [20] and [22] for extremal and equilibrium problem of logarithmic potential with application to the weak asymptotics of polynomial sequences as well as to monograph [7] for discrete orthogonality and constrained equilibrium.

The content of this paper starts with the study of weak asymptotics for monic polynomial sequences (P_n) orthogonal with respect to discrete measures (1.1). In particular, the polynomial sequence orthogonal with respect to negative binomial distribution (Pascal distribution) on \mathbb{N} , i.e.

$$\rho_k = \frac{(\beta)_k}{k!} c^k \quad (\beta > 0, 0 < c < 1), \quad x_k = k, \quad k \in \mathbb{N}, \quad (1.2)$$

is considered. These orthogonal polynomials (denoted by $M_n(x; \beta, c)$) are called *classical Meixner polynomials* [17]. They satisfy the orthogonality conditions

$$\sum_{k=0}^{+\infty} M_n(k; \beta, c) (-k)_j \frac{(\beta)_k}{k!} c^k = 0, \quad j = 0, 1, \dots, n-1, \quad (1.3)$$

where $(-k)_j = -k(1-k) \cdots (j-k-1)$, $j \in \mathbb{N}$, and $(k)_0 = 1$, is the Pochhammer symbol (for the asymptotics of these classical polynomials $M_n(x; \beta, c)$, we refer to [27] and [29]). However, our main goal is the asymptotic analysis of *multiple Meixner polynomials*, which constitute a generalization of the aforementioned classical Meixner polynomials. For these polynomials the orthogonality conditions are considered with respect to a collection of Meixner weights (1.2) with different parameters β or c . The study of these polynomials (among others classical discrete multiple orthogonal polynomials) was initiated in [6].

The structure of the paper is as follows. Section 1 introduces the context and the background materials compressed in four subsections. Indeed, Section 1.1 is devoted to the notion of multiple orthogonal polynomials. In Section 1.2, some general information about recurrence relations for multiple orthogonal polynomials is given. The last two subsections are devoted to some formal properties of the multiple Meixner polynomials obtained in [6]. We recall that there are two kinds of multiple Meixner polynomials. The first kind corresponds (see Section 1.3) to simultaneous orthogonality conditions with respect to discrete weights formed by (1.2) with various c_i , $i = 1, \dots, r$. The second kind polynomials (see Section 1.4) appear when the collection of discrete weights is formed by (1.2) with various β_i , $i = 1, \dots, r$. The explicit expressions for the coefficients of the recurrence relations of multiple Meixner polynomials are the main outcome from Sections 1.3 and 1.4 – used in the sequel.

In Section 2, we state the logarithmic potential equilibrium problems for the description of the weak asymptotics of Meixner and multiple Meixner polynomials. Moreover, the solutions of these problems by means of certain algebraic functions are found. The procedure for solving such type of equilibrium problems is inspired in [2] (see also earlier references therein).

Lastly, in Section 3, by starting from the coefficient of the recurrence relations, we obtain the main term of the asymptotics of multiple Meixner polynomials (the weak asymptotics) and then we check the connection of this term with the equilibrium problem from Section 2. Indeed, Theorems 3.2 and 3.3 are the main results of the present paper. Furthermore, the spectral curves for the multiple Meixner polynomials (of the first and second kind) and their connection with the recurrence relations and vector potential equilibrium with external field and constraint constitute the main outcome of the present approach.

Finally, we would like to highlight that in [23] the weak asymptotics of multiple Meixner polynomials of the second kind were studied, but using a different approach in which the integral representation for the polynomials and the saddle point method were applied as main techniques.

1.1. Multiple orthogonal polynomials (type II)

Below we summarize some results needed for the sequel.

Assume one has r positive measures μ_1, \dots, μ_r on \mathbb{R} , where for each $i = 1, 2, \dots, r$

$$\text{supp}(\mu_i) = \{x \in \mathbb{R}: \forall \epsilon > 0, \mu_i((x - \epsilon, x + \epsilon)) > 0\},$$

denotes the support of μ_i . By Δ_i we denote the smallest interval that contains $\text{supp}(\mu_i)$.

We introduce a multi-index $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ and its length $|\vec{n}| = n_1 + n_2 + \dots + n_r$. Multiple orthogonal polynomials (of type II) can be defined as follows [20, Chapter 4.3], [1] and [28].

Definition 1.1. A type II multiple orthogonal polynomial $P_{\vec{n}}$, corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$, is a polynomial of degree $\leq |\vec{n}|$ which satisfies the orthogonality conditions

$$\int_{\Delta_i} P_{\vec{n}}(x) x^k d\mu_i(x) = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

Here one has a linear system of $|\vec{n}|$ homogeneous relations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}$. Notice that the solution $P_{\vec{n}}$ which is unique up to a multiplicative factor and also that this polynomial solution has exactly degree $|\vec{n}|$ (then the monic multiple orthogonal polynomial exists and will be unique). When this happens \vec{n} is said to be a *normal index* for type II (see [20, second definition on p. 132]).

Suppose that the above r positive measures on \mathbb{R} are discrete, i.e.

$$\mu_i = \sum_{k=0}^{N_i} \rho_{i,k} \delta_{x_{i,k}}, \quad \rho_{i,k} > 0, \quad x_{i,k} \in \mathbb{R}, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, \dots, r,$$

with all the $x_{i,k}$, $k = 0, \dots, N_i$, different and this for each i . In this case we have that $\text{supp}(\mu_i)$ is the closure of $\{x_{i,k}\}_{k=0}^{N_i}$ and that Δ_i is the smallest closed interval on \mathbb{R} which contains $\{x_{i,k}\}_{k=0}^{N_i}$. The corresponding polynomials are then discrete multiple orthogonal polynomials [6]. The discrete measures in this paper will be supported on \mathbb{N} or a subset, which is achieved by taking $x_{i,k} = k$ for $i = 1, \dots, r$. The orthogonality conditions are then more conveniently expressed in terms of the polynomials $(-x)_j$.

Definition 1.2. A discrete multiple orthogonal polynomial of type II on the linear lattice, corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$, is a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}|$ that satisfies the orthogonality conditions

$$\sum_{k=0}^{N_i} P_{\vec{n}}(k) (-k)_j \rho_{i,k} = 0, \quad j = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

In what follows we will focus on a system of measures

$$\mu_i = \sum_{k=0}^N \rho_{i,k} \delta_{x_k}, \quad \rho_{i,k} > 0, \quad x_k \in \mathbb{R}, \quad N \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, \dots, r,$$

where $\text{supp}(\mu_i)$ is the closure of $\{x_k\}_{k=0}^N$ and $\Delta_i = \Delta$ for each $i = 1, \dots, r$, for which every multi-index is normal, namely an *AT system* (see Nikishin and Sorokin [20, p. 140] as well as Arvesú et al. [6, p. 26]). An interesting property related to the location of the zeros of the discrete multiple orthogonal polynomials is given by the following theorem.

Theorem 1.1. (See [6, p. 26].) Suppose we have an AT system of r positive discrete measures. Then every discrete multiple orthogonal polynomial $P_{\vec{n}}$ of type II, corresponding to the multi-index \vec{n} with $|\vec{n}| < N + 1$, has exactly $|\vec{n}|$ different zeros on Δ .

Observe that in an AT system every multi-index \vec{n} , with $|\vec{n}| < N + 1$, is normal. We refer to [6, p. 26] for the following two examples of Chebyshev systems:

Example 1.1. (See [20, Example 3 on p. 138].) The functions

$$\begin{aligned} &v(x)c_1^x, \quad v(x)xc_1^x, \quad \dots, \quad v(x)x^{n_1-1}c_1^x, \\ &\vdots \\ &v(x)c_r^x, \quad v(x)xc_r^x, \quad \dots, \quad v(x)x^{n_r-1}c_r^x, \end{aligned}$$

with all the $c_i > 0$, $i = 1, \dots, r$, different and v a function which has no zeros on \mathbb{R}^+ , form a Chebyshev system on \mathbb{R}^+ for every $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$.

Example 1.2. The functions

$$\begin{aligned} &v(x)\Gamma(x + \beta_1), \quad v(x)x\Gamma(x + \beta_1), \quad \dots, \quad v(x)x^{n_1-1}\Gamma(x + \beta_1), \\ &\vdots \\ &v(x)\Gamma(x + \beta_r), \quad v(x)x\Gamma(x + \beta_r), \quad \dots, \quad v(x)x^{n_r-1}\Gamma(x + \beta_r), \end{aligned}$$

with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$ and v a function with no zeros on \mathbb{R}^+ , form a Chebyshev system on \mathbb{R}^+ for every $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$. If $\beta_i - \beta_j \notin \{0, 1, \dots, N-1\}$ then this still gives a Chebyshev system for every $\vec{n} = (n_1, \dots, n_r)$ for which $n_i < N+1$, $i = 1, 2, \dots, r$.

1.2. Recurrence relation

Suppose that all the multi-indices are normal for the r measures μ_1, \dots, μ_r . Then there exists an interesting recurrence relation of order $r+1$ for the monic multiple orthogonal polynomials of type II with nearly diagonal multi-indices [28]. Here the nearly diagonal multi-index, corresponding to n , is given by

$$\vec{s}(n) = (\underbrace{k+1, k+1, \dots, k+1}_{s \text{ times}}, \underbrace{k, k, \dots, k}_{r-s \text{ times}}),$$

with $n = kr + s$, $0 \leq s < r$. If we write $P_n(x) = P_{\vec{s}(n)}(x)$, then the following recurrence relation holds:

$$xP_n(x) = P_{n+1}(x) + \sum_{j=0}^r a_{n,j} P_{n-j}(x), \quad (1.4)$$

with initial conditions $P_0 = 1$ and $P_j = 0$, $j = -1, -2, \dots, -r$.

A more general form of this recurrence relation is given by

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},0}P_{\vec{n}}(x) + \sum_{j=1}^r b_{\vec{n},j}P_{\vec{n}-\vec{v}_j}(x), \quad (1.5)$$

where \vec{e}_i is the i th standard unit vector in \mathbb{R}^r and $\vec{v}_j = \sum_{k=0}^{j-1} \vec{e}_{r-k}$. In the case $r=2$ the recurrence relation for the polynomials with nearly diagonal multi-indices (1.4) gives the relations $(n+1, n) \rightarrow (n, n) \rightarrow (n, n-1) \rightarrow (n-1, n-1)$ and $(n+1, n+1) \rightarrow (n+1, n) \rightarrow (n, n) \rightarrow (n, n-1)$. The first relation follows from the general case $(n_1+1, n_2) \rightarrow (n_1, n_2) \rightarrow (n_1, n_2-1) \rightarrow (n_1-1, n_2-1)$ by setting $n_1 = n_2 = n$. To obtain the second one we set $n_1 = n$ and $n_2 = n+1$ and interchange the measures μ_1 and μ_2 .

1.3. Multiple Meixner polynomials (first kind)

Let μ_1, \dots, μ_r be defined as

$$\mu_i = \sum_{k=0}^{+\infty} \frac{(\beta)_k c_i^k}{k!} \delta_k, \quad 0 < c_i < 1, \quad i = 1, \dots, r, \quad (1.6)$$

with all the parameters c_i different. The support of these measures is \mathbb{N} and we have $\Delta_1 = \dots = \Delta_r = \mathbb{R}^+$. If $\beta \notin \mathbb{N} \setminus \{0\}$, we define the functions w_i^β , $i = 1, \dots, r$, in $C^\infty(\mathbb{R} \setminus \{-\beta, -\beta-1, \dots\})$ as

$$w_i^\beta(x) = \begin{cases} \frac{\Gamma(\beta+x)}{\Gamma(\beta)} \frac{c_i^x}{\Gamma(x+1)} & \text{if } x \in \mathbb{R} \setminus (\{-1, -2, -3, \dots\} \cup \{-\beta, -\beta-1, -\beta-2, \dots\}), \\ 0 & \text{if } x \in \{-1, -2, -3, \dots\}, \end{cases} \quad (1.7)$$

with simple poles at $-\beta, -\beta-1, -\beta-2, \dots$. If $\beta \in \mathbb{N} \setminus \{0\}$ we define

$$w_i^\beta(x) = \frac{(x+1)_{\beta-1}}{(\beta-1)!} c_i^x,$$

which are functions in C^∞ . These functions can be written as $w_i^\beta(k) = (\beta)_k c_i^k / k!$, $k \in \mathbb{N}$, $i = 1, \dots, r$. By Example 1.1 we know that the measures μ_1, \dots, μ_r form an AT system, which guarantees that every multi-index $\vec{n} = (n_1, \dots, n_r)$ is normal for these measures.

The monic multiple Meixner polynomials of first kind, corresponding to the multi-index $\vec{n} = (n_1, \dots, n_r)$ and the parameters $\beta, \vec{c} = (c_1, \dots, c_r)$, is the unique monic polynomial $M_{\vec{n}}^{\beta; \vec{c}}$ of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\beta; \vec{c}}(k)(-k)_j w_i^{\beta}(k) = 0, \quad j = 0, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

In [6] the following Rodrigues formula was found

$$M_{\vec{n}}^{\beta; \vec{c}}(x) = \prod_{k=1}^r \left(\frac{c_k}{c_k - 1} \right)^{n_k} (\beta)_{|\vec{n}|} \frac{\Gamma(\beta) \Gamma(x+1)}{\Gamma(\beta+x)} \prod_{i=1}^r \left(\frac{1}{c_i^x} \nabla^{n_i} c_i^x \right) \frac{\Gamma(\beta + |\vec{n}| + x)}{\Gamma(\beta + |\vec{n}|) \Gamma(x+1)}.$$

In particular, for $r = 2$, some straightforward calculations yield

$$\begin{aligned} M_{n_1, n_2}^{\beta; c_1, c_2}(x) &= \left(\frac{c_1}{c_1 - 1} \right)^{n_1} \left(\frac{c_2}{c_2 - 1} \right)^{n_2} (\beta)_{n_1+n_2} F_1 \left(-x, -n_1, -n_2; \beta; 1 - \frac{1}{c_1}, 1 - \frac{1}{c_2} \right) \\ &= \frac{c_1^{n_1} c_2^{n_2} (\beta)_{n_1+n_2}}{(c_1 - 1)^{n_1} (c_2 - 1)^{n_2}} \sum_{j=0}^{n_1+n_2} \sum_{k=0}^j \frac{(-n_1)_k (-n_2)_{j-k}}{(\beta)_j} \frac{(\frac{c_1-1}{c_1})^k}{k!} \frac{(\frac{c_2-1}{c_2})^{j-k}}{(j-k)!} (-x)_j, \end{aligned}$$

where

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

is the first of Appell's hypergeometric functions of two variables [10]. From this explicit expression can be found the coefficients of the recurrence relation.

Theorem 1.2. (See [6].) Multiple Meixner polynomials (first kind) $M_{n_1, n_2}^{\beta; c_1, c_2}$ satisfy the following recurrence relations

$$x M_{n_1, n_2}^{\beta; c_1, c_2}(x) = M_{n_1+1, n_2}^{\beta; c_1, c_2}(x) + b_{n_1, n_2} M_{n_1, n_2}^{\beta; c_1, c_2}(x) + c_{n_1, n_2} M_{n_1, n_2-1}^{\beta; c_1, c_2}(x) + d_{n_1, n_2} M_{n_1-1, n_2-1}^{\beta; c_1, c_2}(x),$$

where coefficients are

$$\begin{aligned} b_{n_1, n_2} &= n_1(2a_1 + 1) + n_2(a_1 + a_2 + 1) + a_1\beta, \\ c_{n_1, n_2} &= (n_1(a_1^2 + a_1) + n_2(a_2^2 + a_2))(n_1 + n_2 + \beta - 1), \\ d_{n_1, n_2} &= (\beta + n_1 + n_2 - 1)(\beta + n_1 + n_2 - 2)(a_1 + 1)(a_1 - a_2)a_1n_1, \end{aligned} \tag{1.8}$$

with $a_1 = \frac{c_1}{1-c_1}$ and $a_2 = \frac{c_2}{1-c_2}$.

1.4. Multiple Meixner polynomials (second kind)

Assume that the measures μ_1, \dots, μ_r are given by

$$\mu_i = \sum_{k=0}^{+\infty} \frac{(\beta_i)_k c^k}{k!} \delta_k, \quad \beta_i > 0, \quad i = 1, \dots, r, \tag{1.9}$$

with $0 < c < 1$ and all the β_i different. The support of these measures is \mathbb{N} . In [6] was found that every multi-index $\vec{n} = (n_1, \dots, n_r)$ is normal for the above system of measures μ_1, \dots, μ_r , whenever $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$.

Here

$$w^{\beta_i}(x) = \begin{cases} \frac{\Gamma(\beta_i+x)}{\Gamma(\beta_i)} \frac{c^x}{\Gamma(x+1)} & \text{if } x \in \mathbb{R} \setminus (\{-1, -2, -3, \dots\} \cup \{-\beta, -\beta-1, -\beta-2, \dots\}), \\ 0 & \text{if } x \in \{-1, -2, -3, \dots\}. \end{cases} \tag{1.10}$$

The monic multiple Meixner polynomials of second kind, corresponding to the multi-index \vec{n} and the parameters $\vec{\beta} = (\beta_1, \dots, \beta_r)$, $\beta_i > 0$ ($\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$) and $0 < c < 1$, is the unique monic polynomial $M_{\vec{n}}^{\vec{\beta}; c}$ of degree $|\vec{n}|$ that satisfies the orthogonality conditions

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\vec{\beta}; c}(k)(-k)_j w^{\beta_i}(k) = 0, \quad j = 0, \dots, n_i - 1, \quad i = 1, 2, \dots, r.$$

For this multiple orthogonal polynomial the Rodrigues formula is (see [6])

$$M_{\vec{n}}^{\vec{\beta}; c}(x) = \left(\frac{c}{c-1} \right)^{|\vec{n}|} \prod_{k=1}^r (\beta_i)_{n_i} \frac{\Gamma(x+1)}{c^x} \prod_{i=1}^r \left(\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)} \nabla^{n_i} \frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)} \right) \frac{c^x}{\Gamma(x+1)}.$$

In particular, for $r = 2$ one gets

$$\begin{aligned} M_{n_1, n_2}^{\beta_1, \beta_2; c}(x) &= \left(\frac{c}{c-1} \right)^{n_1+n_2} (\beta_2)_{n_2} (\beta_1)_{n_1} F_{1:0;1}^{1:1;2} \left(\begin{matrix} (-x) : (-n_1); (-n_2, \beta_1 + n_1); \\ (\beta_1) : -; (\beta_2); \end{matrix} ; \frac{c-1}{c}, \frac{c-1}{c} \right) \\ &= \left(\frac{c}{c-1} \right)^{n_1+n_2} (\beta_2)_{n_2} (\beta_1)_{n_1} \sum_{j=0}^{n_1+n_2} \sum_{k=0}^j \frac{(-n_1)_k (-n_2)_{j-k} (\beta_1 + n_1)_{j-k} \left(\frac{c-1}{c} \right)^j}{k!(j-k)! (\beta_2)_{j-k}} \frac{(\frac{c-1}{c})^j}{(\beta_1)_j} (-x)^j, \end{aligned}$$

where

$$F_{l;m;n}^{p;q;k} \left(\begin{matrix} \vec{a} : \vec{b}; \vec{c}; \\ \vec{\alpha} : \vec{\beta}; \vec{\gamma}; \end{matrix} ; x, y \right) = \sum_{r,s=0}^{+\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^l (\alpha_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!},$$

with $\vec{a} = (a_1, \dots, a_p)$, $\vec{b} = (b_1, \dots, b_q)$, $\vec{c} = (c_1, \dots, c_k)$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_l)$, $\vec{\beta} = (\beta_1, \dots, \beta_m)$, $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ are the generalizations of the Kampé de Fériet's series [24] which are a generalization of the four Appell series in two variables. From this explicit expression can be found the coefficients of the recurrence relation.

Theorem 1.3. (See [6].) Multiple Meixner polynomials (second kind) $M_{n_1, n_2}^{\beta_1, \beta_2; c}$ satisfy the following recurrence relations

$$x M_{n_1, n_2}^{\beta_1, \beta_2; c}(x) = M_{n_1+1, n_2}^{\beta_1, \beta_2; c}(x) + b_{n_1, n_2} M_{n_1, n_2}^{\beta_1, \beta_2; c}(x) + c_{n_1, n_2} M_{n_1, n_2-1}^{\beta_1, \beta_2; c}(x) + d_{n_1, n_2} M_{n_1-1, n_2-1}^{\beta_1, \beta_2; c}(x),$$

where coefficients are

$$\begin{aligned} b_{n_1, n_2} &= n_1(2a+1) + n_2(a+1) + a\beta_1, \\ c_{n_1, n_2} &= a(a+1)(n_1 n_2 + n_1(n_1 + \beta_1 - 1) + n_2(n_2 + \beta_2 - 1)), \\ d_{n_1, n_2} &= a^2(a+1)n_1(n_1 + \beta_1 - 1)(n_1 + \beta_1 - \beta_2), \end{aligned} \tag{1.11}$$

with $a = \frac{c}{1-c}$.

2. Equilibrium relations and algebraic functions

Here, inspired in [21] and [9] we introduce the main notions of the approach (based on potential theory) to the weak asymptotics of the polynomials $P_n(x)$, orthogonal with respect to a discrete measure (1.1). Usually one starts with scaling of the problem, i.e.

$$x \rightarrow t: \quad x = x(t, n)$$

(for example, $x = tn$) such that the zeros of the polynomial sequence $\tilde{P}(t) := P_n(x(t, n))$, remain in a compact set $t \in K \Subset \mathbb{C}$ when $n \rightarrow \infty$. This scaling produces a dependence of the orthogonality weight on the parameter n (varying weight):

$$\rho_n(t) = \rho(x(t, n)) =: \exp\{-2nV(t) + O(1)\}.$$

Accordingly, a dependence on n also occurs for its discrete support

$$\{t_{k,n}\}: \quad x_k = x_k(t_{k,n}, n), \quad k \in \mathbb{N},$$

for which one can define a counting measure for the mass points of the support

$$\tau_n(t) := \frac{1}{n} \sum_k \delta(t - t_{k,n}),$$

with the weak limit of this measure

$$\tau(t) = \lim_n \tau_n(t).$$

Under some technical conditions (for details see [9] and [21]) the limiting measure

$$\nu_{P_n}(x(t, n)) \rightarrow \lambda(t),$$

minimizes the following energy functional (with external field V)

$$\mathcal{E}(\mu) := \int (\mathcal{P}^\mu(t) + 2V(t)) d\mu(t), \tag{2.1}$$

among measures from the class

$$\mu: |\mu| = 1, \quad \mu \leq \tau. \quad (2.2)$$

The condition of boundedness of the measure μ by measure τ reflects an interesting peculiarity of the discrete orthogonality – due to the interlacing of masses of the discrete measure and zeros of the polynomials.

The potential of the extremal measure λ satisfies the equilibrium relations with the presence of the external field

$$\mathcal{P}^\lambda(t) + V(t) \begin{cases} \leq \kappa, & t \in \text{supp } \lambda, \\ = \kappa, & t \in \text{supp}(\tau - \lambda) \cap \text{supp}(\lambda) := \Sigma, \\ \geq \kappa, & t \in \text{supp}(\tau - \lambda). \end{cases} \quad (2.3)$$

Set Σ , where the combination of the potential and the external field, is equal to a constant is called *equilibrium zone*. A set where the equilibrium measure λ touches its constraint, i.e. measure τ , is called *saturation zone*. As it follows from equilibrium relations (2.3) in the saturation zone the combination of the potential and the external field is strictly less than the equilibrium constant κ .

To describe the weak asymptotics of multiple orthogonal polynomials an important notion of *vector potential equilibrium* was introduced by Gonchar and Rakhmanov in [11] and [12].

Let $\tilde{\Delta} := \{\Delta_1, \dots, \Delta_r\}$ be a collection of compact sets in \mathbb{C} and let

$$D = (d_{i,j})_{i,j=1}^r$$

be a real symmetric nonsingular positive definite matrix. An additional condition on D to be compatible with $(\Delta_1, \dots, \Delta_r)$ is that $d_{i,j} \geq 0$ whenever $\Delta_i \cap \Delta_j \neq \emptyset$.

For a vector of measures

$$\vec{\mu} = (\mu_1, \dots, \mu_r), \quad \text{supp } \mu_j \subset \Delta_j, \quad j = 1, \dots, r,$$

the energy functional $I(\vec{\mu})$ is defined as

$$I(\vec{\mu}) = \sum_{i=1}^r \sum_{j=1}^r d_{i,j} I(\mu_i, \mu_j), \quad (2.4)$$

where $I(\mu_i, \mu_j)$ is the mutual energy of two scalar measures

$$I(\mu_i, \mu_j) = \int_{\Delta_i} \int_{\Delta_j} \ln \frac{1}{|x - t|} d\mu_i(x) d\mu_j(t).$$

The extremal vector measure $\vec{\lambda}$, minimizing the energy functional (2.4) among all $\vec{\mu}$ (with fixed masses of the coordinates) possesses the equilibrium properties

$$U_j^{\vec{\lambda}}(x) = \sum_{i=1}^r d_{ij} \mathcal{P}^{\lambda_i}(x) \begin{cases} = \kappa_j, & x \in \text{supp } \lambda_j = \Delta_j^*, \\ \geq \kappa_j, & x \in \Delta_j \setminus \Delta_j^*. \end{cases} \quad \text{for } j = 1, \dots, r. \quad (2.5)$$

Here the vector $\vec{U}^{\vec{\lambda}} = (U_1^{\vec{\lambda}}, \dots, U_r^{\vec{\lambda}})$ is called *vector potential of the vector valued measure $\vec{\lambda}$ with respect to the interaction matrix D* .

External fields $\vec{V} = (V_1, \dots, V_r)$ such that $V_j(t)$ is a function on $t \in \Delta_j$, $j = 1, \dots, r$, and constraining for $\vec{\mu}$ measures as follows

$$\vec{\tau} = (\tau_1, \dots, \tau_r): \quad \mu_j \leq \tau_j, \quad \text{supp } \tau_j \subset \Delta_j, \quad j = 1, \dots, r,$$

can be incorporated in the vector equilibrium problem in order to develop an approach for multiple orthogonality with respect to varying weights or (and) with respect to discrete weights (in a similar fashion that [21] and [9]). Thus, the energy functional (2.1) of vector measure with external fields becomes (see (2.4))

$$\mathcal{E}(\vec{\mu}) := I(\vec{\mu}) + 2 \sum_{j=1}^r \int V_j(t) d\mu_j(t),$$

and extremal vector measure $\vec{\lambda}$, minimizing this energy functional possesses the equilibrium properties (see (2.5)) for $j = 1, \dots, r$

$$U_j^{\vec{\lambda}}(x) + V_j(x) \begin{cases} \leq \kappa_j, & x \in \text{supp } \lambda_j, \\ = \kappa_j, & x \in \text{supp}(\tau_j - \lambda_j) \cap \text{supp } \lambda_j = \Sigma_j, \\ \geq \kappa_j, & x \in \text{supp}(\tau_j - \lambda_j). \end{cases} \quad (2.6)$$

A saddle point extremal problem for functional (2.6) is considered in some applications (when positions of the components of $\vec{\Delta}$ are not known). In such a situation one performs the minimization of the energy functional by measure $\vec{\mu}$ and maximizes it by vector $\vec{\Delta}$. As a result the support of the equilibrium measure possesses the following symmetry relations

$$\frac{\partial(U_j + V_j)}{\partial n_+} = \frac{\partial(U_j + V_j)}{\partial n_-}, \quad \text{on } \Sigma_j, \quad j = 1, \dots, r, \quad (2.7)$$

where $\partial/\partial n_{\pm}$ denotes the normal derivatives on the respective contours. This property is called S-symmetry after Stahl, who introduced it in [25] and [26].

To conclude this short survey on the logarithmic potential approach to the weak asymptotics we recall one general class of multiple orthogonal polynomials introduced by Nikishin in [18]. Such systems are defined as follows. Let σ_1, σ_2 be two finite Borel measures with constant sign, whose supports $\text{supp } \sigma_1, \text{supp } \sigma_2$ are contained in nonintersecting intervals Δ_1, Δ_2 , respectively, of the real line \mathbb{R} . Set

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t} d\sigma_1(x).$$

This expression defines a new measure whose support coincides with that of σ_1 . In general, let $\vec{\sigma} = (\sigma_1, \dots, \sigma_r)$ be a system of finite Borel measures on the real line with constant sign and compact support. Let $\Delta_k = [a_k, b_k]$ denote the smallest interval which contains the support of σ_k . Assume that $\Delta_k \cap \Delta_{k+1} = \emptyset, k = 1, \dots, m-1$. We say that $\vec{S} = (s_1, \dots, s_r) = \mathcal{N}(\vec{\sigma})$ is the *Nikishin system* generated by $\vec{\sigma}$, when

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \quad \dots, \quad s_r = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_r \rangle \rangle. \quad (2.8)$$

In what follows (without loss of generality) we assume $r = 2$. So if we introduce the following notation for the Markov function (Cauchy transform)

$$\hat{s}(z) := \int \frac{ds(x)}{z-x},$$

then Nikishin system of two measures is: $s_1 = \sigma_1, s_2 = \hat{\sigma}_2 \sigma_1$. These measures have the same support. The Nikishin system is a useful model for multiple orthogonal polynomials with respect to a collection of measures (μ_1, μ_2) with the same support ($\Delta := \text{supp } \mu_1 = \text{supp } \mu_2$). For such type of systems an important role plays the analytic continuation of the function

$$\mathcal{F} := \frac{\widehat{\mu}_{2+} - \widehat{\mu}_{2-}}{\widehat{\mu}_{1+} - \widehat{\mu}_{1-}}, \quad \text{on } \Delta. \quad (2.9)$$

For the pair (μ_1, μ_2) forming a Nikishin system (2.8), $(\mu_1, \mu_2) := (s_1, s_2)$ we have

$$\mathcal{F} = \hat{\sigma}_2 \in H(\bar{\mathbb{C}} \setminus \Delta_2).$$

The limiting zero counting measure for multiple orthogonal polynomial $P_{\vec{n}}, \vec{n} := (n, n)$ with respect to a Nikishin system of measures (s_1, s_2)

$$\nu_{P_n} \rightarrow \lambda,$$

is described by equilibrium problem (2.5) with the matrix of interaction

$$D := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (2.10)$$

and $\lambda = \lambda_1$, where $|\lambda_1| = 2, \text{supp } \lambda_1 \in \Delta_1$ and $|\lambda_2| = 1, \text{supp } \lambda_2 \in \Delta_2$. Here the measure λ_2 is the limiting zero counting measure

$$\nu_{R_n^{(j)}} \rightarrow \lambda_2, \quad j = 1, 2,$$

for the functions of the second kind

$$R_n^{(j)} := \int \frac{P_n(x) ds_j(x)}{z-x}, \quad j = 1, 2.$$

Since the original contribution [19] by Nikishin there have been published numerous papers devoted to the asymptotics of multiple orthogonal polynomial with respect to a Nikishin system of measures (for most general results and latest surveys see [4] and [3]).

In this section, starting from the fact that components of the vector of discrete Meixner weights have the same support we assume that these measures inherit properties of the Nikishin system of continuous measures with the same support (which is not developed for the discrete measures yet). This assumption allows us heuristically to state an ad hoc vector

equilibrium problem with Nikishin matrix of interaction, with external field (because of unbounded set of orthogonality) and constraint (due to discrete weight). Accordingly, at this stage we look for the solutions of this equilibrium problem in a form of spectral curve (among the simplest candidates satisfying the necessary conditions following from the equilibrium problem). Thus, as a result of the first stage of our analysis we have heuristically got equations of the algebraic curves (2.27) and (2.41) and have related their logarithms (rigorously) with certain vector equilibrium problems (see Propositions 2.1 and 2.2). We highlight that at this point spectral curves and equilibrium problems are not formally related to the multiple Meixner orthogonal polynomials.

In Section 3 we derive the same spectral curves from the recurrence coefficients of the multiple Meixner orthogonal polynomials. Consequently, it then rigorously relates Nikishin type equilibrium with constraints to Meixner discrete orthogonality.

2.1. Potential problem and its solution for classical Meixner polynomials

In this subsection, aimed to illustrate our approach in connection with the above discussed techniques, we take (as example) the classical Meixner polynomials (1.3). First, we state the equilibrium problem for the limiting zero counting measure. Then, we show our approach, which is based on an algebraic function formulation, by solving this equilibrium problem in a quite direct way.

The scaling of variable x in (1.2)

$$x =: nt$$

stabilizes the zeros of the polynomials (i.e. when $n \rightarrow \infty$, the zeros stay in a compact)

$$\tilde{P}_n(t) := M_n(x),$$

which one considers now as a polynomial of variable t . Indeed, it is orthogonal to $\{t^k\}_{k=0}^{n-1}$ with respect to the varying weight (depending on n)

$$\tilde{\rho}_n(t) := \frac{\Gamma(\beta + tn)}{(tn)!} c^{tn},$$

concentrated in the points $\{\frac{k}{n}\}_{k \in \mathbb{N}}$. Taking the limit

$$\lim_{n \rightarrow \infty} |\tilde{\rho}_n(t)|^{1/n} = c^t = \exp \left\{ -t \ln \frac{1}{c} \right\},$$

we see that the external field $V(t)$ involved in the energy functional (2.1) and in the equilibrium relations (2.3) has the following form

$$V(t) = \ln \frac{1}{c} \operatorname{Re} t. \quad (2.11)$$

We also have

$$\frac{1}{n} \sum_{k=0}^{\infty} \delta \left(t - \frac{k}{n} \right) \xrightarrow{*} l(t) =: \tau(t), \quad t \in \mathbb{R}_+, \quad (2.12)$$

where $l(t)$ is the Lebesgue measure with unit density (dt) on the semiaxis \mathbb{R}_+ .

Thus, the limiting measure $\lambda(t)$ of zero distributions of polynomials $\tilde{P}_n(t)$

$$\nu_{\tilde{P}_n(t)} \xrightarrow{*} \lambda(t),$$

satisfy the relations (2.1)–(2.2) as well as the conditions of equilibrium (2.3) with external field (2.11) and constraint (2.12).

Let us show our approach for finding the explicit representation of the measure $\lambda(t)$. We look for the Cauchy transform of measure λ (or Markov function)

$$h(z) := \hat{\lambda}(z) = \int \frac{d\lambda(t)}{z - t}, \quad z \in \mathbb{C} \setminus \operatorname{supp} \lambda, \quad (2.13)$$

as function $h(z)$, which be analytic on some two sheeted Riemann surface $\mathcal{R} := (\mathbb{C}^{(0)}, \mathbb{C}^{(1)})$. The possibility of analytic continuation of the function $h(z)$ to another sheet of the Riemann surface through the compact Σ can be seen from the second relation in (2.3) after it “complexification” and differentiation. Thus, the function $h := (h_0, h_1)$ on \mathcal{R} has some natural condition for its values at ∞ . For the main branch h_0 , because of (2.13) (and normalization of measure $|\lambda| = 1$)

$$h(z) := \frac{1}{z} + \dots, \quad z \rightarrow \infty^{(0)}, \quad (2.14)$$

and for its continuation h_1 (taking into account derivative of the “complexification” of external field (2.11))

$$h(z) := \ln \frac{1}{c} - \frac{1}{z} + \dots, \quad z \rightarrow \infty^{(1)}. \quad (2.15)$$

The density of the measure $\lambda'(t)$ is proportional to the jump of imaginary part of the function $h(z)$ along \mathbb{R}_+ . Moreover, it is clear from electrostatic consideration of the relations (2.1)–(2.2) that at the neighborhood of the left end point of $\text{supp } \lambda$, i.e. to the right from the origin, this jump has to be a constant – here the measure λ reaches its constraint by the Lebesgue measure $l(t)$. Such behavior can be modelled by representing the function $h(z)$ as logarithm of some other function, say $E(z)$, which on this part of the support of the measure λ (i.e. in the saturation zone) takes negative real values. Hence, we are looking for $h(z)$ in the form

$$h(z) = \ln E(z), \quad (2.16)$$

where $E(z)$ is a rational function on \mathcal{R} with one zero and one pole (at the points on the different sheets of Riemann surface \mathcal{R} with the same projection point 0)

$$E(z) = \begin{cases} 0, & z = 0^{(0)}, \\ \infty, & z = 0^{(1)}. \end{cases} \quad (2.17)$$

We have chosen the position of these points in order to provide holomorphicity of branches h – Eq. (2.16) in $\mathbb{C} \setminus \mathbb{R}_+$. The conditions (2.14) and (2.15) give the following behavior for $E(z)$ in the neighborhood of infinity on both sheets of \mathcal{R}

$$E(z) \sim \begin{cases} E_0(z) = 1 + \frac{1}{z}, & z \rightarrow \infty^{(0)}, \\ E_1(z) = \frac{1}{c}(1 - \frac{1}{z}), & z \rightarrow \infty^{(1)}. \end{cases} \quad (2.18)$$

Observe that it is enough to have conditions (2.17) and (2.18) for determining $E(z)$ as well as the equation of its Riemann surface. Moreover, the product $E_0(z)E_1(z)$ behaves like $1/c$ at infinity, then by Liouville's theorem this product is precisely $1/c$ (since no singularities). In addition, $E_0(z) + E_1(z)$ will behave like $\alpha/z + \beta$, which is the simplest form of a function that has a pole at 0 (here $\beta = -c^{-1} - 1$ and $\alpha = c^{-1} - 1$). Indeed, the Vieta relations give

$$E^2 + \left[\frac{1}{z} \left(\frac{1}{c} - 1 \right) - \left(\frac{1}{c} + 1 \right) \right] E + \frac{1}{c} = 0.$$

From here it follows that Riemann surface of the function $E(z)$ has branch points

$$e_1 = \frac{1 - \sqrt{c}}{1 + \sqrt{c}}, \quad e_2 = \frac{1}{e_1}.$$

Moreover, both branches of the function $E(z)$ are negative on $[0, e_1]$ and positive on $[e_2, \infty]$. Therefore, the jump of the imaginary part of the function $h(z)$ – see Eq. (2.16) – is a constant on $[0, e_1]$ and equal zero on $[e_2, \infty]$. Thus, $\text{supp } \lambda = [0, e_2]$, the interval $[0, e_1]$ is a saturation zone, the interval $[e_1, e_2] = \Sigma$ corresponds to the equilibrium zone, and $[e_2, \infty]$ is a zone free from zeros of the scaled classical Meixner polynomials. Measure λ obtained by this procedure satisfies the equilibrium condition (2.3). Furthermore, after integration one can obtain an explicit value of the equilibrium constant in (2.3)

$$\kappa = 1 + \ln \frac{1 - c}{c}.$$

2.2. Potential problem and its solution for multiple Meixner polynomials (first kind)

Assume $r = 2$, and $\vec{n} = (n, n)$. We rewrite weights (1.7) as follows

$$w_j^\beta(x) = \frac{\Gamma(\beta + x)}{x!} e^{x \ln c_j}, \quad j = 1, 2.$$

Here the supports of both discrete orthogonality measures coincide. Therefore, we pose an equilibrium problem for the weak asymptotics in a similar way to the Nikishin case. Indeed, we consider the equilibrium problem associated with the weights $w_1^\beta(x)$, and $u(x)w_1^\beta(x)$, where

$$u(x) = \frac{w_2^\beta(x)}{w_1^\beta(x)} = e^{x \ln \frac{c_2}{c_1}} = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{t \ln \frac{c_2}{c_1}}}{t - x} dt, \quad c_2 < c_1.$$

The integral representation in the right hand side is due to the Cauchy integral formula with a contour of integration Γ encircling \mathbb{R}_+ , i.e. it comes from infinity through the upper half plane along the positive semiaxis turning counterclockwise

around zero in the left half plane and returning to infinity along the positive semiaxis through the lower half plane. In addition, the contour Γ serves as a boundary of holomorphicity region of an analog of the function (2.9).

Now we do a scaling of the problem in order to compactify zeros of multiple Meixner polynomials (first kind). Let

$$\tilde{M}_{2n}^{\beta;\tilde{c}}(t) = n^{-2n} M_{n,n}^{\beta;\tilde{c}}(tn) = t^{2n} + \dots \quad (2.19)$$

Recall that the function of the second kind associated with the multiple Meixner polynomials of first kind is given by

$$R_n^{(j)}(y) = \int \frac{M_{n,n}^{\beta;\tilde{c}}(x)}{x-y} d\mu_j(x), \quad j = 1, 2,$$

where μ_j is the discrete orthogonality measure (1.6). Let us denote $\tilde{R}_n^{(j)}(z) = R_n^{(j)}(nz)$. General feature of the Nikishin systems is that the functions of the second kind have n -zeros on a compact of the complex plane outside of the support of the orthogonality measures. For the classical Nikishin system (see (2.8)) they accumulate along interval Δ_2 . For the case under consideration we expect that the function $\tilde{R}_n^{(j)}(z)$ has n -zeros accumulated along an analytic arc γ in the complex plane. We assume that contour Γ contains this analytic arc γ .

For the formulation of the equilibrium problem, we follow the Nikishin system model, in which the zeros accumulate along an open contour $\gamma \subset \mathbb{C}$ (for the functions of the second kind) and in \mathbb{R}_+ (for the multiple orthogonal polynomials). Since the mass points x_k are uniformly distributed over a subset of \mathbb{R} , then the zero counting measure $\nu_{\tilde{M}_{2n}^{\beta;\tilde{c}}} \xrightarrow{*} \lambda$, where $|\lambda| = 2$, and $\lambda \leq l$, being $dl(t) = dt$ the Lebesgue measure, and $\text{supp}(\lambda) \subset \mathbb{R}_+$. On the other hand, $\nu_{\tilde{R}_n^{(j)}} \xrightarrow{*} \mu$, and its magnitude is equal to 1 with $\text{supp}(\mu) \subset \gamma$. Accordingly,

$$\begin{cases} \lambda: & |\lambda| = 2, \quad \lambda \leq l, \quad \text{supp}(\lambda) \subset \mathbb{R}_+, \\ \mu: & |\mu| = 1, \quad \text{supp}(\mu) =: \gamma \subset \Gamma. \end{cases} \quad (2.20)$$

Now we write the equilibrium problem for these measures $(\lambda, \mu) := \tilde{\lambda}$ as (2.6), where:

- The matrix of interaction D in the definition (2.5) of the vector potential $U_j^{\tilde{\lambda}}$ has the Nikishin's form (2.10).
- Scaling of the weights w_1^β and u brings external fields $V_1 := (-\ln c_1) \text{Re } t$ on \mathbb{R}_+ and $V_2 := (-\ln \frac{c_2}{c_1}) \text{Re } t$ on Γ , respectively.

Thus, we have the equilibrium relations:

$$\begin{cases} W_1 := 2\mathcal{P}^\lambda(t) - \mathcal{P}^\mu(t) - (\ln c_1) \text{Re } t & \begin{cases} \leq \kappa_1, & \text{on } \text{supp}(\lambda) \subset \mathbb{R}_+, \\ \geq \kappa_1, & \text{on } \mathbb{R}_+ \cap \text{supp}(l \setminus \lambda), \end{cases} \\ W_2 := 2\mathcal{P}^\mu(t) - \mathcal{P}^\lambda(t) - (\ln \frac{c_2}{c_1}) \text{Re } t & \begin{cases} = \kappa_2, & \text{on } \text{supp}(\mu) =: \gamma \subset \Gamma, \\ \geq \kappa_2, & \text{on } \Gamma. \end{cases} \end{cases} \quad (2.21)$$

We also have to add to this system the S-symmetry property (2.7) for the second plate Γ of our condenser (\mathbb{R}_+, Γ) :

$$\frac{\partial W_2}{\partial n_+} = \frac{\partial W_2}{\partial n_-}, \quad \text{on } \gamma. \quad (2.22)$$

Now we look for a solution of the equilibrium problem (2.20), (2.21), (2.22). In accordance with our approach (by differentiating) we pass from (2.21) to the following relations for the real part of the Cauchy transforms of the measures λ and μ ,

$$\begin{aligned} \text{Re}\{2\hat{\lambda} - \hat{\mu} + \ln c_1\} &= 0, & \text{on } \text{supp}(\lambda) \cap \text{supp}(l \setminus \lambda) &=: \Sigma, \\ \text{Re}\{2\hat{\mu} - \hat{\lambda} + \ln c_2 - \ln c_1\} &= 0, & \text{on } \text{supp}(\mu) &:= \gamma, \end{aligned}$$

respectively. From these expressions one rewrites

$$\begin{aligned} \hat{\lambda} &= \hat{\mu} - \hat{\lambda} - \ln c_1, & \text{on } \Sigma, \\ \hat{\mu} + \ln c_2 &= \hat{\lambda} - \hat{\mu} + \ln c_1, & \text{on } \gamma. \end{aligned} \quad (2.23)$$

Notice that if we define on the three sheeted Riemann surface

$$\mathcal{R} := (\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2): \quad \begin{cases} \mathcal{R}_0 := \tilde{\mathbb{C}} \setminus \Sigma, \\ \mathcal{R}_1 := \tilde{\mathbb{C}} \setminus \{\Sigma \cup \gamma\}, \\ \mathcal{R}_2 := \tilde{\mathbb{C}} \setminus \gamma, \end{cases}$$

a function

$$H(z) := \begin{cases} \hat{\lambda}, & \text{in } \mathcal{R}_0, \\ \hat{\mu} - \hat{\lambda} - \ln c_1, & \text{in } \mathcal{R}_1, \\ -\hat{\mu} - \ln c_2, & \text{in } \mathcal{R}_2, \end{cases}$$

which has a jump on $\text{supp}(\lambda) \cap \mathcal{R}_0$, then due to (2.23) and (2.22) the branches of this function have analytic continuation one to another through the cuts joining the sheets $(\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2)$ of the Riemann surface \mathcal{R} . Observe that from (2.20) follows that $H(z)$ behaves near infinity as

$$H(z) = \begin{cases} 2z^{-1} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_0, \\ -\ln c_1 - z^{-1} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_1, \\ -\ln c_2 - z^{-1} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_2. \end{cases}$$

Similarly, like we proceeded for the classical Meixner polynomials we model the constraint condition $\lambda \leq l$ and $dl(t) = dt$ on \mathbb{R}_+ , representing the function $H(z)$ as logarithm of a meromorphic function $\phi(z)$ on \mathcal{R} , i.e.

$$H(z) = \ln \phi(z), \quad \phi \in \mathcal{M}(\mathcal{R}). \quad (2.24)$$

To have the jump for the function H along \mathbb{R}_+ lifted to sheet \mathcal{R}_0 we force the function $\phi(z)$ to take values $\phi = \infty$, and $\phi = 0$ at the points from the two sheets $(\mathcal{R}_0, \mathcal{R}_1)$ which have on \mathbb{C} projection point $z = 0$

$$\phi(z) = \begin{cases} 0, & \text{as } z \rightarrow 0 \text{ in } \mathcal{R}_0, \\ \infty, & \text{as } z \rightarrow 0 \text{ in } \mathcal{R}_1. \end{cases} \quad (2.25)$$

Thus, $H \in \mathcal{M}(\mathcal{R} \setminus \{\mathbb{R}_+^{(0)} \cup \mathbb{R}_+^{(1)}\})$. From the behavior of H at infinity we conclude that

$$\phi(z) = \begin{cases} 1 + 2z^{-1} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_0, \\ c_1^{-1}(1 - z^{-1}) + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_1, \\ c_2^{-1}(1 - z^{-1}) + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_2. \end{cases} \quad (2.26)$$

If Σ and γ are connected then the genus of the Riemann surface \mathcal{R} is equal to zero (we assume that it is true). The simplest meromorphic function ϕ (i.e. see (2.25) – one pole and one zero) that maps \mathcal{R} to the Riemann sphere and therefore the inverse function $z(\phi)$ is just a rational function. From (2.26) we know the poles z as a function of ϕ and moreover we know its residues in the poles. Taking into account also (2.25) one gets that the algebraic function $\phi(z)$ is defined by the equation

$$\phi(z): \quad z = \frac{2}{\phi - 1} - \frac{1}{c_1\phi - 1} - \frac{1}{c_2\phi - 1}. \quad (2.27)$$

This algebraic function has four branch points. In order to construct (for the condenser with two plates (\mathbb{R}_+, Γ) , $\Gamma \subset \mathbb{C} \setminus \mathbb{R}_+$) a solution of the equilibrium problem (2.21)–(2.22) we need to have the case when:

- ϕ has two complex conjugated branch points e, \bar{e} , and two real e_1, e_2 ($0 < e_1 < e_2$);
- the analytic arc γ joining points (e, \bar{e}) , chosen by means of (2.22), does not cross \mathbb{R}_+ ;
- values $\phi(\zeta) \in \mathbb{R}_-$, when $\zeta \in [0, e_1]^{(0)} \subset \mathcal{R}_0$.

The first item can be checked by analyzing the critical values c_1, c_2 when the branch points of the function ϕ coincide – the transition of two conjugate branch points to the case of two real branch points. For this purpose we take consequently two times the discriminant of the curve (2.27) – first time in the variable ϕ and second time in the variable z . Hence, we get a polynomial of two variables c_1 and c_2 with rational coefficients. Factorization of this polynomial gives a factor of high degree (maximal degree in each variable c_j is 6 and maximal degree of the monomials is 9). It defines an algebraic curve $T(c_1, c_2) = 0$, which has genus equal 1. Thus, this curve is rather complicated. However, taking into account the symmetry of this curve with respect to (c_1, c_2) , and changing variables

$$\begin{cases} c_1 + c_2 =: s, \\ c_1 c_2 =: p, \end{cases}$$

yields the algebraic curve

$$\tilde{T}(s, p) := T(c_1(s, p), c_2(s, p)) = 0, \quad (2.28)$$

which has genus equal 0. Therefore, curve $\tilde{T}(s, p)$ admits a rational uniformization. Using Gröbner basis technique we arrive to the following simple parametrization of this curve

$$\begin{cases} s = -\frac{(\tau^2 + 8\tau + 18)(\tau + 2)^2}{2\tau}, \\ p = -\frac{(\tau + 3)^3(\tau + 2)^3}{\tau^3}. \end{cases}$$

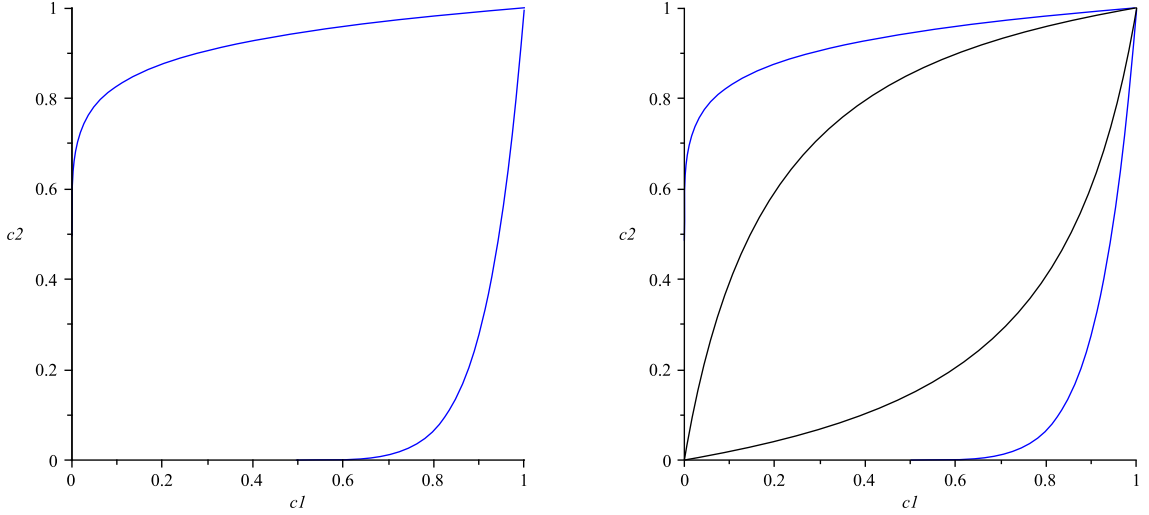


Fig. 1. (A) – on the left. Domain \mathcal{GN} . If $(c_1, c_2) \in \mathcal{GN}$, then ϕ has two complex and two real branch points; (B) – on the right. Domain $\mathcal{N} \subset \mathcal{GN}$. If $(c_1, c_2) \in \mathcal{N}$ then $\gamma \cap \mathbb{R}_+ = \emptyset$.

An analysis of this curve shows that $c_1, c_2 \in \mathbb{R}$ when $\tau \in [-\infty, -2] \cap [0, \infty]$. In Fig. 1(A) for $\tau \in [-4.73, -3]$ we see branches of the critical curve (2.28) which belong to the interesting for us quadrate $(c_1, c_2) \in \{(0, 1) \times (0, 1)\}$. If parameters (c_1, c_2) are inside of the domain bounded by these branches and axes $\{c_1 = 0, c_2 \in (0, 0.5)\}$ and $\{c_2 = 0, c_1 \in (0, 0.5)\}$ then function ϕ has two complex conjugated branch points e, \bar{e} , and two real e_1, e_2 ($0 < e_1 < e_2$). We denote this domain \mathcal{GN} . On the critical lines the conjugate branch points e, \bar{e} coincide on the real axes and when $(c_1, c_2) \in \{(0, 1) \times (0, 1)\} \setminus \mathcal{GN} =: \mathcal{A}$, the function ϕ has four real branch points.

The second item, i.e. condition on the parameters (c_1, c_2) implying $\gamma \cap \mathbb{R}_+ = \emptyset$ can be clarified using an equivalent (constructive) definition of the S-curve γ (see (2.22)). It is known (see [15, 25]) that S-property condition (2.22) means that the analytic curve γ belongs to the union of the critical (orthogonal) trajectories of the quadratic differentials

$$\operatorname{Re} \int (H_j - H_k) dz = 0 \iff (H_j - H_k)^2 (dz)^2 < 0. \quad (2.29)$$

Using uniformization (2.24), (2.27) we perform integration by parts

$$\int H(z) dz = z \ln \phi - \int \frac{z}{\phi} d\phi = z \ln \phi + \frac{(\phi - 1)^2}{(\phi - \sigma_1)(\phi - \sigma_2)}, \quad \sigma_j := 1/c_j, \quad j = 1, 2.$$

Thus condition (2.29) is equivalent to

$$\left| \exp \left\{ \int (H_j - H_k) dz \right\} \right| = 1 \iff \left| \left(\frac{\phi_j}{\phi_k} \right)^z \frac{(\phi_j - 1)^2 (\phi_k - \sigma_1)(\phi_k - \sigma_2)}{(\phi_k - 1)^2 (\phi_j - \sigma_1)(\phi_j - \sigma_2)} \right| = 1. \quad (2.30)$$

Now we obtain the critical values of the parameters such that curve γ is crossing the origin. Substituting in (2.30) $j = 1, k = 2, z = 0$ and taking from (2.25), (2.27) value of $\phi(0)$ we have

$$\left| \frac{(\phi_2(0) - \sigma_1)(\phi_2(0) - \sigma_2)}{(\phi_2(0) - 1)^2} \right| = 1, \quad \text{and} \quad \phi_2(0) = \frac{2\sigma_1\sigma_2 - \sigma_1 - \sigma_2}{\sigma_1 + \sigma_2 - 2}.$$

Therefore, taking into account that $\sigma_1, \sigma_2 \in \mathbb{R}$ we arrive to

$$\left| \frac{(\sigma_1 - 1)(\sigma_2 - 1)}{(\sigma_1 - \sigma_2)^2} \right| = \frac{1}{4}, \quad \text{or} \quad \frac{(\sigma_1 + \sigma_2 - 2)^2 - (\sigma_1 - \sigma_2)^2}{(\sigma_1 - \sigma_2)^2} = \pm 1.$$

A branch of the critical curve corresponding to sign $(-)$ in the right hand side lies out of the admissible range for c_1, c_2 (or σ_1, σ_2). Finally, considering in the right hand side sign $(+)$ we get

$$(\sigma_1 + \sigma_2 - 2)^2 = 2(\sigma_1 - \sigma_2)^2. \quad (2.31)$$

In Fig. 1(B) the branches of the critical curve (2.31) forms the boundary of the domain \mathcal{N} . In this domain $(c_1, c_2) \in \mathcal{N}$ the analytic arc γ does not intersect \mathbb{R}_+ (see Fig. 2(A)). On the boundary of this domain γ is passing trough the origin (see Fig. 2(B)).

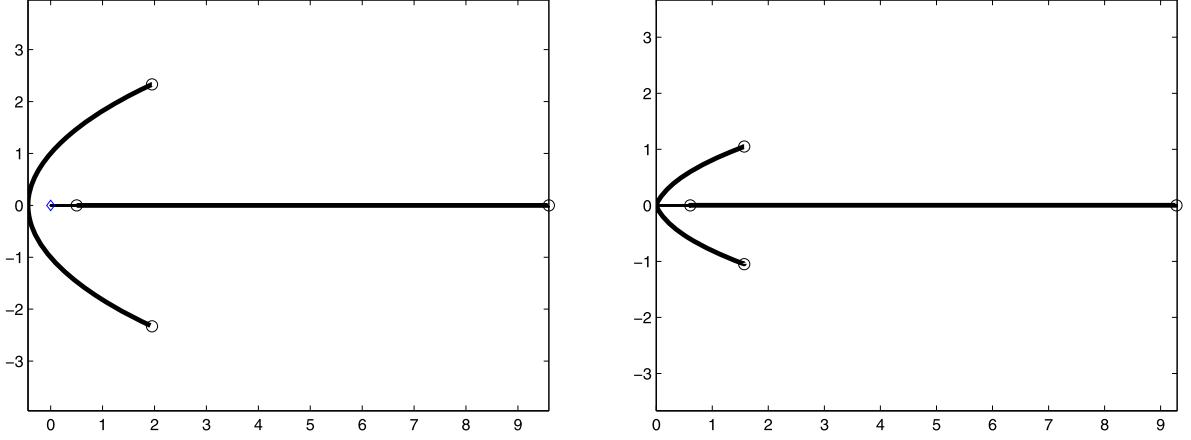


Fig. 2. The sets $\gamma = \text{supp}(\mu)$ and $[0, e_2] = \text{supp}(\lambda)$ as well as the branch points (circles) are depicted for the cases: (A) – on the left. $(c_1, c_2) = (0.5, 0.25) \in \mathcal{N}$; (B) – on the right. $(c_1, c_2) = (0.5, 0.146445649)$ is critical.

The cut joining the logarithmic singularities of the function H starts at point 0 on sheet \mathcal{R}_0 , goes along \mathbb{R}_+ to branch point e_1 , passing to sheet \mathcal{R}_1 and goes there back to point 0. Along this cut (for all $(c_1, c_2) \in \{(0, 1) \times (0, 1)\}$) values ϕ are negative. Thus we have, indeed

$$H \in \mathcal{M}(\mathcal{R} \setminus \{[0, e_1]^{(0)} \cup [e_1, 0]^{(1)}\}).$$

Summarizing we have that for the parameters belonging to the domain

$$\mathcal{N} := \left\{ (c_1, c_2) \in \{(0, 1) \times (0, 1)\} : \frac{c_2(1 - \sqrt{2})}{2c_2 - 1 - \sqrt{2}} < c_1 < \frac{c_2(1 + \sqrt{2})}{2c_2 - 1 + \sqrt{2}} \right\}, \quad (2.32)$$

the density of measure λ is determined by the limiting values at the points $[0, e_2]$ of the imaginary part of $\ln \phi(z)/\pi$ or equivalently,

$$\frac{d\lambda(t)}{dt} = \lim_{\tilde{z} \rightarrow 0} \frac{1}{\pi} \text{Im} H(t + i\tilde{z}), \quad t \in [0, e_2] \text{ in } \mathcal{R}_0. \quad (2.33)$$

The interval $[0, e_2]$ is the support of the measure λ . Since the function ϕ is negative on $[0, e_1]$ in \mathcal{R}_0 we have that the density of the measure λ is equal to unity there. Therefore, the interval $[0, e_1]$ corresponds to the saturation zone and the interval $\Sigma = [e_1, e_2]$ is the equilibrium zone. The support of measure μ is the analytic arc γ . Thus, we have proved the following result.

Proposition 2.1. *The density of an absolutely continuous measure λ , solving equilibrium problem (2.20), (2.21), (2.22) for $(c_1, c_2) \in \mathcal{N}$, is equal to (2.33), where the function H is defined by (2.24) and (2.27).*

Notice that the present paper addresses the existence of a solution of vector equilibrium problem with constraint while the uniqueness remains out of the scope of this paper. We refer to [8] and [13] on this question for some specific vector equilibrium problem with constraint.

Remark 2.1. We recall that for classical Nikishin system (on two disjoint intervals in \mathbb{R}) the asymptotic zero distribution of the polynomials is recovered as one component of the solution of the corresponding vector equilibrium problem. In general, for Nikishin system formed with two disjoint contours in the complex plane satisfying S-symmetry, it is not proved yet. Though, for special Nikishin systems in the complex plane it can be proved by performing an ‘ad hoc’ analysis (as it is done in [15,16], and in this paper).

Remark 2.2. In Section 3.2, we prove that the logarithmic potential of the measure λ defined in (2.33) is equal to the exponent of the main term of the asymptotics of the multiple Meixner polynomials for all values of parameters $(c_1, c_2) \in \{(0, 1) \times (0, 1)\}$ (see Theorem 3.3). Therefore, for all values of the parameters, the zero counting measure of the polynomials converges to λ

$$\nu_{\tilde{M}_{2n}^{\beta; \tilde{c}}} \xrightarrow{*} \lambda,$$

and the interval between the origin and the smallest real branch point will be the saturation zone. However, for $(c_1, c_2) \in \{(0, 1) \times (0, 1)\} \setminus \mathcal{N}$ the measure λ is not any more a solution of the equilibrium problem (2.20), (2.21), (2.22) with Nikishin’s

matrix of interaction. Without going into details, we just mention that measure λ from (2.33) for $(c_1, c_2) \in \mathcal{A}$ can be presented as a solution of an equilibrium problem with Angelesco matrix of interaction (see [11]). For $(c_1, c_2) \in \mathcal{GN} \setminus \mathcal{N}$ it will be more sophisticated equilibrium problem for the condenser with three plates (see examples in [3] and [4]). It is interesting to mention, that if we take the multiple Laguerre polynomials considered in [15,16] as the direct continuous analogue of the multiple Meixner polynomials we will see a complete correspondence of these three asymptotical regimes $(\mathcal{N}, \mathcal{A}$ and $\mathcal{GN} \setminus \mathcal{N})$ between the continuous and discrete case.

2.3. Potential problem and its solution for multiple Meixner polynomials (second kind)

The algebraic equations and the equilibrium problems studied here for the multiple Meixner polynomials of the second kind were already addressed in [23], but using another approach.

Let $r = 2$ and $0 < \beta_2 < \beta_1 < \beta_2 + 1$ in the representation (1.10) for weights of the discrete orthogonality measures. Now writing the measures of orthogonality for multiple Meixner polynomials (second kind) in form of a Nikishin system, we consider the ratio of the weights

$$\hat{u}(x) = \frac{w^{\beta_2}(x)}{w^{\beta_1}(x)} = \sum_{k=0}^{\infty} \frac{v_k}{x - x_k}, \quad v_k = \frac{1}{\Gamma(\beta_1 - \beta_2)} \frac{(\beta_2 - \beta_1 + 1)_k}{k!}, \quad x_k = -\beta_2 - k.$$

This meromorphic function can be understood as the Cauchy transform of a discrete measure u with support belonging to \mathbb{R}_- . Thus orthogonality measures of multiple Meixner polynomials (second kind) form a Nikishin system generated by two discrete measures: μ_1 (see (1.9)) with support in \mathbb{R}_+ and u with support in \mathbb{R}_- . Since the mass points for both measures are uniformly distributed over a subset of \mathbb{R}_+ and \mathbb{R}_- , respectively, then the limiting zero counting measures in the equilibrium problem will be constrained by $dl(x) = dx$ the Lebesgue measure.

Doing a scaling

$$\tilde{M}_{2n}^{\beta_1, \beta_2; c}(t) := M_{n,n}^{\beta_1, \beta_2; c}(nt), \quad \tilde{R}_n^{(\beta_1, \beta_2; c), j}(z) = R_n^{(\beta_1, \beta_2; c), j}(nz), \quad j = 1, 2, \quad (2.34)$$

we get the varying weight of orthogonality which defines an external field on \mathbb{R}_+ .

Thus, we expect that the zero counting measures associated with the scaled multiple orthogonal polynomial and the functions of the second kind weakly converge

$$\nu_{\tilde{M}_{2n}^{\beta_1, \beta_2; c}} \xrightarrow{*} \lambda, \quad \nu_{\tilde{R}_n^{(\beta_1, \beta_2; c), j}} \xrightarrow{*} \mu, \quad j = 1, 2,$$

to measures $(\lambda, \mu) := \vec{\lambda}$

$$\begin{cases} |\lambda| = 2, & \lambda \leq l, & \text{supp}(\lambda) \subset \mathbb{R}_+, \\ |\mu| = 1, & \mu \leq l, & \text{supp}(\mu) \subset \mathbb{R}_-. \end{cases} \quad (2.35)$$

Now we write the equilibrium problem for these measures $\vec{\lambda} = (\lambda, \mu)$ as (2.6), where:

- Matrix of interaction D in the definition (2.5) of the vector potential $U_j^{\vec{\lambda}}$ has the Nikishin's form (2.10).
- Scaling of the weights w^{β_1} brings external fields $V_1 := (-\ln c) \text{Re } t$ on \mathbb{R}_+ .

Thus, we have the equilibrium relations:

$$\begin{cases} W_1 := 2\mathcal{P}^\lambda - \mathcal{P}^\mu - \ln c \text{Re } t & \begin{cases} \leq \kappa_1, & \text{on } \text{supp}(\lambda) \subset \mathbb{R}_+, \\ \geq \kappa_1, & \text{on } \mathbb{R}_+ \cap \text{supp}(l \setminus \lambda), \end{cases} \\ W_2 := 2\mathcal{P}^\mu - \mathcal{P}^\lambda & \begin{cases} \leq \kappa_2, & \text{on } \text{supp}(\mu) \subset \mathbb{R}_-, \\ \geq \kappa_2, & \text{on } \mathbb{R}_- \cap \text{supp}(l \setminus \mu). \end{cases} \end{cases} \quad (2.36)$$

Now we look for a solution of the equilibrium problem (2.35), (2.36). In accordance with our approach (by differentiating) we pass from (2.36) to the following relations for the Cauchy transforms of the measures λ and μ ,

$$\begin{aligned} \hat{\lambda} &= \hat{\mu} - \hat{\lambda} - \ln c, & \text{on } \text{supp}(\lambda) \cap \text{supp}(l \setminus \lambda) &=: \Sigma_\lambda, \\ \hat{\mu} &= \hat{\lambda} - \hat{\mu}, & \text{on } \text{supp}(\mu) \cap \text{supp}(l \setminus \mu) &=: \Sigma_\mu. \end{aligned} \quad (2.37)$$

We point out that if we define on the three sheeted Riemann surface

$$\mathcal{R} := (\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2): \quad \begin{cases} \mathcal{R}_0 := \tilde{\mathbb{C}} \setminus \Sigma_\lambda, \\ \mathcal{R}_1 := \tilde{\mathbb{C}} \setminus \{\Sigma_\lambda \cup \Sigma_\mu\}, \\ \mathcal{R}_2 := \tilde{\mathbb{C}} \setminus \Sigma_\mu \end{cases}$$

a function

$$H(z) := \begin{cases} \hat{\lambda}, & \text{in } \mathcal{R}_0, \\ \hat{\mu} - \hat{\lambda} - \ln c, & \text{in } \mathcal{R}_1, \\ -\hat{\mu} - \ln c, & \text{in } \mathcal{R}_2, \end{cases}$$

then due to (2.37) the branches of this function have analytic continuation one to another throw the cuts joining the sheets $(\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2)$ of the Riemann surface \mathcal{R} . Observe that from (2.35) follows that $H(z)$ behaves near infinity as

$$H(z) = \begin{cases} \frac{z}{2} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_0, \\ -\ln c - \frac{1}{z} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_1 \text{ and } \mathcal{R}_2. \end{cases}$$

In the same way like we did it in the previous subsections we model the constraint condition ($\lambda \leq l$ on \mathbb{R}_+ and $\mu \leq l$ on \mathbb{R}_- , $dl(x) = dx$) representing the function $H(z)$ as logarithm of a meromorphic function $\phi(z)$ on \mathcal{R} , i.e.

$$H(z) = \ln \phi(z), \quad \phi \in \mathcal{M}(\mathcal{R}). \quad (2.38)$$

To have a jump for the function H along \mathbb{R}_+ lifted to sheet \mathcal{R}_0 and along \mathbb{R}_- lifted to sheet \mathcal{R}_1 we force the function $\phi(z)$ to have pole $\phi = \infty$ of order 2 at the point $z = 0$ on the sheet \mathcal{R}_1 , and zeros $\phi = 0$ of order 1 at the points $z = 0$ on the two sheets \mathcal{R}_0 and \mathcal{R}_2

$$\phi(z) = \begin{cases} 0, & \text{as } z \rightarrow 0 \text{ in } \mathcal{R}_0, \\ \infty^2, & \text{as } z \rightarrow 0 \text{ in } \mathcal{R}_1, \\ 0 & \text{as } z \rightarrow 0 \text{ in } \mathcal{R}_2. \end{cases} \quad (2.39)$$

Hence, $H \in \mathcal{M}(\mathcal{R} \setminus \{\mathbb{R}_+^{(0)} \cup \mathbb{R}^{(1)} \cup \mathbb{R}_-^{(2)}\})$. From the behavior of H at infinity we conclude

$$\phi(z) = \begin{cases} 1 + \frac{z}{2} + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_0, \\ \frac{1}{c}(1 - \frac{1}{z}) + \dots, & \text{as } z \rightarrow \infty \text{ in } \mathcal{R}_1 \text{ and } \mathcal{R}_2. \end{cases} \quad (2.40)$$

Now we obtain an equation for the algebraic function ϕ . We write expansions of branches ϕ up to $\mathcal{O}(1/z^3)$

$$\begin{aligned} \phi_0(z) &= 1 + \frac{2}{z} + \frac{A}{z^2} + \mathcal{O}(z^{-3}), \\ \phi_1(z) &= \sigma - \frac{\sigma}{z} + \frac{B}{z^2} + \mathcal{O}(z^{-3}), \quad \text{where } \sigma = c^{-1}, \\ \phi_2(z) &= \sigma - \frac{\sigma}{z} + \frac{D}{z^2} + \mathcal{O}(z^{-3}). \end{aligned}$$

Since ϕ is a rational function on the Riemann surface \mathcal{R} and all its zeros and poles are shown in (2.39), we consider the Vieta relations for the product and for the pairwise products of these expansions and obtain the following linear system for unknown A and $B + D$:

$$\begin{cases} (B + D) + 3\sigma + \sigma A = 0 \\ (1 + \sigma)(B + D) + 4\sigma - \sigma^2 + 2\sigma A = 0 \end{cases} \Rightarrow A := -\frac{4\sigma - 1}{\sigma - 1}, \quad B + D := \frac{\sigma^2 + 2\sigma}{\sigma - 1}.$$

Hence, we obtain

$$\phi^3 - \frac{(2x^2 - 2x - 1)\sigma + (x + 1)^2}{x^2} \phi^2 + \sigma \frac{(x - 2)\sigma + 2(x + 1)}{x} \phi - \sigma^2 = 0. \quad (2.41)$$

This algebraic function has four branch points, namely infinity point ∞ and three real points e_-, e_1, e_2 ($e_- < 0 < e_1 < e_2$). The cut joining the logarithmic singularities of the function H starts at point 0 on sheet \mathcal{R}_0 goes along \mathbb{R}_+ to the branch point e_1 , passing to sheet \mathcal{R}_1 goes there back via point 0 to the branch point e_- and then again passing to sheet \mathcal{R}_2 goes there back to point 0. Thus we have, indeed

$$H \in \mathcal{M}(\mathcal{R} \setminus \{[0, e_1]^{(0)} \cup [e_1, e_-]^{(1)} \cup [e_-, 0]^{(2)}\}).$$

Recall that the density of the measure λ is determined by the limiting values at the points $[0, e_2]$ of the imaginary part of the $\ln \phi(z)/\pi$ or equivalently,

$$\frac{d\lambda(t)}{dt} = \lim_{y \rightarrow 0} \frac{1}{\pi} \operatorname{Im} H(t + iy), \quad t \in [0, e_2] \text{ in } \mathcal{R}_0. \quad (2.42)$$

Since function ϕ is negative on $[0, e_1]$ in \mathcal{R}_0 we have that density of the measure λ is equal unity there. Therefore, $[0, e_1]$ corresponds to the saturation zone and $\Sigma = [e_1, e_2]$ is the equilibrium zone. Thus, we have proved the following result.

Proposition 2.2. *The density of the absolutely continuous measure λ , solving equilibrium problem (2.35), (2.36) is equal to (2.42), where the function H is defined by (2.38) and (2.41).*

3. The n th-root asymptotic from recurrence relations and equilibrium problem

In this section, starting from the coefficients of the recurrence relations, we obtain the main term of the asymptotics of multiple Meixner polynomials and then we check the connection of this term with the equilibrium problem studied in the previous section.

An important ingredient of this approach is the expression of the main term of the asymptotics for the solutions of the difference equation which coefficients depending on a parameter, i.e. recurrence relation with 'varying' coefficients. For the three-term recurrence relations with varying coefficients such an expression has been obtained by Kuijlaars and Van Assche in [14]. The multiple Meixner polynomials (studied in this paper) satisfy a four-term recurrence relation (1.8), (1.11) with varying coefficients after the scaling (2.19), (2.34). An asymptotic expression for the multiple orthogonal polynomial solutions of the four-term recurrence relation with varying coefficients (similar to those of [14]) has been recently given in [5]. Below we apply this asymptotic formula for the multiple Meixner polynomials.

The main steps of our approach here are the following.

1) We begin by forming a transition matrix for our recursion. We consider vectors \vec{V}_1 , \vec{V}_2 and \vec{V}_3 with coordinates

$$\frac{M_{n_1, n_2}(x)}{n_1! n_2!}, \quad (3.1)$$

taken for the following values of the multi-index (n_1, n_2) :

$$\begin{aligned} [(n, n); (n, n-1); (n-1, n-1)]^T &\longrightarrow \vec{V}_1, \\ [(n+1, n); (n, n); (n, n-1)]^T &\longrightarrow \vec{V}_2, \\ [(n+1, n+1); (n+1, n); (n, n)]^T &\longrightarrow \vec{V}_3. \end{aligned}$$

We define transition matrices $A_n^{(j)}$, $j = 1, 2$, by means of relations:

$$A_n^{(1)} \vec{V}_1 = \vec{V}_2, \quad A_n^{(2)} \vec{V}_2 = \vec{V}_3.$$

To obtain the coefficients of these transition matrices we need two types of recurrence relations for M_{n_1, n_2} , which connects the following indices

$$\begin{cases} I) & (n, n) \text{ and } (n+1, n), (n, n), (n, n-1), (n-1, n-1), \\ II) & (n+1, n) \text{ and } (n+1, n+1), (n+1, n), (n, n), (n, n-1), \end{cases} \quad (3.2)$$

as it was already indicated in (1.5). Indeed, the first type of the recursions we already have. They follow from (1.8) and (1.11) if we set $(n_1, n_2) = (n, n)$. To get the recursions of the second type we substitute $(n_1, n_2) = (n, n+1)$ in (1.8) and (1.11) and then interchange a_1 with a_2 and β_1 with β_2 , respectively.

2) The analysis of recursions for polynomials (3.1) with indices (3.2) (these recursions follows from (1.8) and (1.11)) shows that if we change the variable there as

$$\tilde{t} := \frac{x}{n},$$

then we get the recurrence relations with coefficients having limits when $n \rightarrow \infty$. It gives us a regime

$$\begin{cases} n \rightarrow \infty, \\ \tilde{t} \in K \subseteq \mathbb{C}, \end{cases} \quad (3.3)$$

for the investigation of asymptotics starting from the recurrence relations. Thus we form a limiting transition matrix

$$A(t) = \lim_{n \rightarrow \infty} A_n^{(2)} A_n^{(1)}, \quad t \in K \subseteq \mathbb{C}, \quad (3.4)$$

and find eigenvalues of this matrix. We fix the following order for these eigenvalues

$$|L_1(\tilde{t})| \geq |L_2(\tilde{t})| \geq |L_3(\tilde{t})|, \quad \tilde{t} \in \mathbb{C}.$$

3) Then, n th root asymptotics for polynomials (3.1) can be obtained by means of the following formula proven in [5] (we remark that our notation $L_1(\tilde{t})$ corresponds to the notation $\lambda_1(t)$ from [5] as follows $L_1(\tilde{t}) = \lambda_1(t)$ and $\tilde{t} = \frac{1}{t}$).

Proposition 3.1. (See [5, Theorem 3.1].) *There holds the following asymptotic formula (when $n \rightarrow \infty$)*

$$\frac{1}{n} \ln \left| \frac{M_{n,n}(x)}{(n!)^2} \right| = \frac{1}{n} \int_0^n \ln \left| L_1 \left(\frac{x}{\tilde{n}} \right) \right| d\tilde{n} + o(1), \quad (3.5)$$

uniformly for $\frac{x}{n} \in K \subseteq \Omega$, where Ω is a domain containing ∞ -point and bounded by a curve $\Gamma := \{\tilde{t}: |L_1(\tilde{t})| = |L_2(\tilde{t})|\}$.

We note that function $L(\tilde{t})$ is an algebraic function since it is a root of the polynomials in L with rational coefficients in \tilde{t} (which is the characteristic polynomial of the matrix $A(\tilde{t})$). This observation allows us to use uniformization of the algebraic curve $L(\tilde{t})$

$$\begin{cases} L = F_1(s), \\ n = \frac{x}{F_2(s)}, \end{cases} \quad \text{where } s \text{ is the parameter of uniformization,}$$

to evaluate explicitly the integral in the right hand side of (3.5).

4) Finally we check that the obtained explicit expression for the right hand side of (3.5) is the logarithmic potential of measure λ , which is a solution of the corresponding vector equilibrium problem from the previous section

$$\frac{1}{n} \int_0^n \ln \left| L_1 \left(\frac{x}{\tilde{n}} \right) \right| d\tilde{n} = \mathcal{P}^\lambda(t).$$

For this purpose we differentiate

$$\frac{d}{dt} \left[\frac{1}{n} \int_0^n \ln \left(L_1 \left(\frac{x}{\tilde{n}} \right) \right) d\tilde{n} \right] \Big|_{n=\frac{x}{t}} =: H(t), \quad (3.6)$$

and we verify that

$$H(t) = \ln \phi_0(t) = \int \frac{d\lambda(\xi)}{t - \xi},$$

where $\phi(t)$ is the algebraic function from the previous section, which gives an explicit solution of the equilibrium problem.

Notice that (3.6) relates the largest eigenvalue L_1 to the branch ϕ_0 of the algebraic function ϕ (see below its algebraic equations (3.8) and (3.9), for multiple Meixner polynomials of the second and first kind, respectively). Similarly, the eigenvalues L_2 and L_3 are related to branches ϕ_1 and ϕ_2 , respectively.

Remark 3.1. In what follows, the parameter a involved in $A^{(1)}$ (multiple Meixner polynomials of the second kind) and the parameters a_1, a_2 involved in $A^{(1)}, A^{(2)}$ (multiple Meixner polynomials of the first kind) come from (1.8) and (1.11) respectively.

3.1. Asymptotics for multiple Meixner polynomials of the second kind

We start with the polynomials of the second kind since the calculations for obtaining the asymptotics look simplest. Indeed, the following result is valid.

Theorem 3.2. *The main term of asymptotics for multiple Meixner orthogonal polynomials of the second kind has the form*

$$\frac{1}{n} \ln \left| \frac{M_{n,n}^{\beta_1, \beta_2, c}(x)}{(n!)^2} \right| = \mathcal{P}^\lambda \left(\frac{n}{x} \right) + o(1),$$

as $n \rightarrow \infty$, uniformly for $\frac{n}{x} \in K \Subset \mathbb{C}$, where at the right hand side stands the logarithmic potential of the equilibrium measure λ for the equilibrium problem (2.35), (2.36). Furthermore, the Cauchy transform of measure λ has the form

$$\hat{\lambda}(t) = H(t) = \ln \phi_0(t), \quad t \in \mathbb{C} \setminus [0, e_2],$$

where $\phi_0(t) = 1 + \frac{t}{e_2}$, $t \rightarrow \infty$ is branch (2.40) of the algebraic function ϕ , defined by Eq. (2.41), and e_2 is the maximal positive branch points of the function ϕ .

Proof. Following the step 1) of our approach (see description at the beginning of this section) we compute the transition matrices $A_n^{(1)}$ and $A_n^{(2)}$, and we consider them in the limit regime (3.3). Doing it we note that the obtained limits coincide. This circumstance allows us to use the matrix

$$A^{(1)}(t) = \lim_{n \rightarrow \infty} A_n^{(1)}, \quad t \in K \Subset \mathbb{C},$$

instead of the transition matrix $A(t)$ from (3.4). We just have to multiply by 2 the right hand side of formula (3.5). Hence, we have

$$A^{(1)}(t) = \begin{pmatrix} \frac{1}{2}t - \frac{3}{2}a - 1 & -\frac{3}{4}a(a+1) & -\frac{1}{8}a^2(a+1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial $P(L, t)$ of the matrix $A^{(1)}(t)$ has the form:

$$P = L^3 - \frac{1}{2}L^2t + \frac{3}{2}L^2a + \frac{3}{4}a^2L + \frac{3}{4}aL + \frac{1}{8}a^3 + \frac{1}{8}a^2.$$

The equation $P(L, t) = 0$ defines an algebraic function $L(t)$ of genus zero, therefore this function allows a rational uniformization. Moreover, taking into account the linear appearance of t in $P(L, t)$ we immediately obtain this uniformization if we take L as the uniformization parameter:

$$\begin{cases} L = L, \\ t = \frac{P(L, t) + \frac{1}{2}L^2t}{\frac{1}{2}L^2} = \frac{1}{4} \frac{(2L + b + b^2)(2L - b + b^2)(2L - 1 + b^2)}{L^2}, \end{cases}$$

where a is changed ($a := b^2 - 1$). From here, since $t = \frac{x}{n}$, we have

$$n = \frac{4L^2x}{(2L + b + b^2)(2L - b + b^2)(2L - 1 + b^2)}. \quad (3.7)$$

Now we can integrate by parts the right hand side of (3.5), i.e.

$$2 \left(\int \ln(L) dn \right) = 2n \ln(L) + x \ln(2L + b + b^2) + x \ln(2L - b + b^2) - 2x \ln(2L - 1 + b^2).$$

Then, we substitute $x(n)$ from (3.7) into the obtained expression and multiplying it by $1/n$, as a result we obtain the right hand side of (3.5)

$$\begin{aligned} \frac{2}{n} \int \ln(L) dn &= 2 \ln(L) + \frac{1}{4} \frac{1}{L^2} \left((2L + b + b^2)(2L - 1 + b^2) \ln \left(\frac{(2L + b + b^2)(2L - b + b^2)}{(2L - 1 + b^2)^2} \right) (2L - b + b^2) \right) \\ &=: F(L). \end{aligned}$$

It remains to check, that the real part of $F(L)$ is indeed the logarithmic potential of the equilibrium measure λ of the problem (2.35), (2.36). It will be true if the derivative of $F(L)$ with respect to t coincides with the function $H(t) = \ln \phi(t)$, where $\phi(t)$ is the algebraic function (2.41). To differentiate F we use the uniformization variable L . Thus,

$$\frac{d}{dt} F(L(t)) = \frac{\frac{d}{dL} F(L)}{\frac{d}{dL} t(L)} = \ln \left(\frac{(2L + b + b^2)(2L - b + b^2)}{(2L - 1 + b^2)^2} \right) =: H.$$

Finally, substituting

$$\begin{aligned} \phi &:= e^H = \frac{(2L + b + b^2)(2L - b + b^2)}{(2L - 1 + b^2)^2}; \\ z &:= t = \frac{1}{4} \frac{(2L + b + b^2)(2L - b + b^2)(2L - 1 + b^2)^2}{L^2}; \\ b &:= \frac{1}{\sqrt{1 - 1/\sigma}}, \quad \sigma = c^{-1}, \end{aligned}$$

we get an equation for the function $\phi(z)$, i.e.

$$z^2 \phi^3 - \phi^2 (2\sigma z^2 + z^2 - 2z\sigma + 2z - \sigma + 1) + z\sigma (z\sigma + 2z - 2\sigma + 2)\phi - \sigma^2 z^2 = 0, \quad (3.8)$$

which coincide with (2.41). Theorem is completely proved. \square

3.2. Asymptotics for multiple Meixner polynomials of the first kind

Finally, we consider the Meixner multiple orthogonal polynomials of the first kind.

Theorem 3.3. *The main term of asymptotics for the multiple Meixner orthogonal polynomials of first kind has the form*

$$\frac{1}{n} \ln \left| \frac{M_{n,n}^{\beta; \vec{c}}(x)}{(n!)^2} \right| = \mathcal{P}^\lambda \left(\frac{n}{x} \right) + o(1),$$

being the convergence uniformly for $\frac{n}{x} \in K \subseteq \mathbb{C}$ and $n \rightarrow \infty$, where at the right hand side stands the logarithmic potential of a measure λ defined by its Cauchy transform which has the form

$$\hat{\lambda}(t) = H(t) = \ln \phi_0(t), \quad t \in \mathbb{C} \setminus [0, e_2],$$

where $\phi_0(t) = 1 + \frac{2}{t}$, $t \rightarrow \infty$ is branch (2.40) of the algebraic function ϕ , defined by Eq. (2.27), and e_2 is the maximal positive branch points of the function ϕ .

For parameters $(c_1, c_2) \in \mathcal{N}$ (see (2.32)) the measure λ is the solution of the equilibrium problem (2.20), (2.21), (2.22).

Proof. Firstly, we compute the transition matrices $A_n^{(1)}$ and $A_n^{(2)}$, and we consider them in the limit regime (3.3)

$$A^{(j)}(t) = \lim_{n \rightarrow \infty} A_n^{(j)}, \quad t := \frac{x}{n} \in K \subseteq \mathbb{C}.$$

For the case of the Meixner multiple orthogonal polynomials of first kind these limiting matrices are different. We have

$$A^{(1)} = \begin{pmatrix} \frac{1}{2}t - \frac{3}{2}a_1 - \frac{1}{2}a_2 - 1 & -\frac{1}{2}a_1(a_1 + 1) - \frac{1}{2}a_2(a_2 + 1) & -\frac{1}{2}a_1(a_1 + 1)(a_1 - a_2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$A^{(2)} = \begin{pmatrix} \frac{1}{2}t - \frac{3}{2}a_2 - \frac{1}{2}a_1 - 1 & -\frac{1}{2}a_1(a_1 + 1) - \frac{1}{2}a_2(a_2 + 1) & -\frac{1}{2}a_2(a_2 + 1)(a_2 - a_1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we compute the characteristic polynomial $P(L, t)$ of the transition matrix – product $A^{(2)}A^{(1)}(t)$. Like in Theorem 3.2 the equation $P(L, t) = 0$ defines an algebraic function $L(t)$ of genus zero. Rational uniformization of this algebraic curve has the following form

$$\begin{cases} L = \frac{(s - a_2(a_2 + 1)(a_1 - a_2))(s + a_1(a_1 + 1)(a_1 - a_2))}{(a_1(a_1 + 1) + a_2(a_2 + 1))^2}, \\ t = \frac{((s + a_1^2(a_1 + 1) + a_2^2(a_2 + 1))(s + a_1(a_1 + 1)^2 + a_2(a_2 + 1)^2)(2s + (a_2 + a_1 + 1)(a_1 - a_2)^2))}{((a_1(a_1 + 1) + a_2(a_2 + 1))(s + a_1(a_1 + 1)(a_1 - a_2))(s - a_2(a_2 + 1)(a_1 - a_2)))}, \end{cases}$$

with s as a parameter of the uniformization. From here, since $n = tx$, we integrate by parts the right hand side of (3.5)

$$\int \ln(L) dn = n \ln(L) - \ln\left(\frac{s + a_1^2 + a_1^3 + a_2^2 + a_2^3}{s + a_1 + 2a_1^2 + a_1^3 + a_2 + 2a_2^2 + a_2^3}\right)x.$$

Hence, multiplying it by $\frac{1}{n}$ we obtain the right hand side of (3.5)

$$\frac{1}{n} \int \ln(L) dn = \ln(L(s)) + \ln\left(\frac{s + a_1^2 + a_1^3 + a_2^2 + a_2^3}{s + a_1 + 2a_1^2 + a_1^3 + a_2 + 2a_2^2 + a_2^3}\right)t(s) =: F(s).$$

It remains to check, that the real part of F is indeed the logarithmic potential of the equilibrium measure λ of the problem (2.20), (2.21), (2.22). Thus, we differentiate F using the uniformization variable s . Accordingly, we have

$$H := \ln(\phi) := \frac{d}{dt}F = \frac{\frac{d}{ds}F}{\frac{d}{ds}t} = -\ln\left(\frac{s + a_1^2 + a_1^3 + a_2^2 + a_2^3}{s + a_1 + 2a_1^2 + a_1^3 + a_2 + 2a_2^2 + a_2^3}\right) =: \ln(r(s)).$$

From here we get a rational uniformization of the curve

$$\phi(z): \quad \phi = r(s), \quad z = t(s),$$

which bring us an equation for the function $\phi(z)$:

$$\begin{aligned} a_2 a_1 z \phi^3 + (-3a_2 a_1 z - z a_1 + a_1 - z a_2 + a_2) \phi^2 + (z - 2 - a_2 - a_1 + 2z a_2 + 2z a_1 + 3a_2 a_1 z) \phi \\ - z(a_2 + 1)(a_1 + 1) = 0, \end{aligned} \tag{3.9}$$

or equivalently

$$z - \frac{2}{\phi - 1} + \frac{a_1 + 1}{a_1 \phi - 1 - a_1} + \frac{a_2 + 1}{a_2 \phi - a_2 - 1} = 0,$$

which after substitution $a_1 = \frac{c_1}{1-c_1}$ and $a_2 = \frac{c_2}{1-c_2}$ coincides with (2.27). It remains to apply Proposition 2.1. The theorem is proved. \square

4. Conclusions and future directions

In this paper we have investigated the n th root asymptotic behavior of the multiple Meixner polynomials of the first and second kind, respectively (see Theorems 3.2, 3.3). We have developed an approach based on an algebraic function formulation in connection with some available techniques from logarithmic potential theory. Indeed, for each polynomial sequence we have posed the corresponding equilibrium problem for the limiting zero counting measure and then we solved it in a quite direct way. An analysis of the asymptotic distribution of zeros of multiple Meixner polynomials was given.

Finally, other systems of discrete multiple orthogonal polynomials (for $r = 2$) might be asymptotically analyzed using the approach presented here (up to some adaptations) to produce similar outcomes.

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